Subgroups of the Group of Self-Homotopy Equivalences

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ABSTRACT. Denote by $\mathcal{E}(Y)$ the group of homotopy classes of self-homotopy equivalences of a finite-dimensional complex Y. We give a selection of results about certain subgroups of $\mathcal{E}(Y)$. We establish a connection between the Gottlieb groups of Y and the subgroup of $\mathcal{E}(Y)$ consisting of homotopy classes of self-homotopy equivalences that fix homotopy groups through the dimension of Y, denoted by $\mathcal{E}_{\#}(Y)$. We give an upper bound for the solvability class of $\mathcal{E}_{\#}(Y)$ in terms of a cone decomposition of Y. We dualize the latter result to obtain an upper bound for the solvability class of the subgroup of $\mathcal{E}(Y)$ consisting of homotopy classes of self-homotopy equivalences that fix cohomology groups with various coefficients. We also show that with integer coefficients, the latter group is nilpotent.

1. Introduction and Preliminaries

Let Y be a CW-complex of dimension N and $\mathcal{E}(Y)$ the group of homotopy classes of self-homotopy equivalences of Y. In this paper we present a sample of results about a number of subgroups of $\mathcal{E}(Y)$. We denote by $\mathcal{E}_{\#}(Y)$ the following proto-typical such subgroup:

$$\mathcal{E}_{\#}(Y) = \{f \in \mathcal{E}(Y) \mid f_{\#} = 1\colon \pi_i(Y) \to \pi_i(Y), \text{ for all } i \leq N\}.$$

In Section 2, we give a way to construct elements in $\mathcal{E}_{\#}(Y)$. This is of interest since it provides a connection between the *Gottlieb group* of Y and certain subgroups of self-homotopy equivalences (Theorem 2.3). Next, in Section 3, we consider questions about the solvability and nilpotency of $\mathcal{E}_{\#}(Y)$. For example, we show that if Y is the cofibre of a map between two wedges of spheres, then $\mathcal{E}_{\#}(Y)$ is an abelian group (Corollary 3.5). This result generalizes into a simple upper bound on the solvability of $\mathcal{E}_{\#}(Y)$ in terms of a cone-length invariant of Y (Theorem 3.3). In Section 4 we dualize these results to obtain upper bounds for the solvability of the group of equivalences that fix cohomology with different coefficients (Theorem 4.2). We also show that the subgroup of self-homotopy equivalences which fix the integral cohomology of a finite complex is a nilpotent group (Proposition 4.9).

We now review some standard material that we will use. A cofibration sequence

$$Z \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma Z,$$

where X is the mapping cone of γ, gives a homotopy coaction $c\colon X \to X \vee \Sigma Z$, obtained by pinching the 'equator' of the cone of Z to a point. The coaction induces an action of $[\Sigma Z, W]$ on $[X, W]$ for any space W (cf. [**Hil65**, Chap. 15] for details). This is defined as follows: If $\alpha \in [\Sigma Z, W]$ and $f \in [X, W]$, then f^{α} is the composition

$$X \xrightarrow{c} X \vee \Sigma Z \xrightarrow{f \vee \alpha} W \vee W \xrightarrow{\nabla} W.$$

The following properties of this action are well-known, and follow easily from the definitions:

1. If $h\colon W \to W'$, then $h(f^{\alpha}) = (hf)^{h\alpha}$.
2. If $\alpha, \beta \in [\Sigma Z, W]$, then $(f^{\alpha})^{\beta} = f^{(\alpha + \beta)}$.

Next, consider the following portion of the Puppe sequence associated to the above cofibration sequence:

$$[\Sigma Z, W] \xrightarrow{q^*} [X, W] \xrightarrow{j^*} [Y, W].$$

As is also well-known, the orbits of the action are precisely the pre-images of j^*. That is, for $f, g \in [X, W]$, we have $f^{\alpha} = g$ for some $\alpha \in [\Sigma Z, W]$ if and only if $fj = gj$.

Next, we review some notation and terminology for groups. Suppose that G is a group and H and K are subgroups. Then $H \lhd G$ denotes that H is normal in G and $[H, K]$ denotes the subgroup generated by commutators of elements of H with elements of K. A *normal chain* for G is a sequence of subgroups

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{k+1} \supseteq \cdots$$

with $G_{i+1} \lhd G_i$ for $i \geq 1$. If $[G_i, G_i] \subseteq G_{i+1}$ for each i, then the sequence is called a *solvability series*. If, further, $G_{k+1} = \{1\}$, then we say that G is *solvable of class* $\leq k$ and write solv $G \leq k$. Analogously, given a normal chain as above with each $G_i \lhd G$ and $[G, G_i] \subseteq G_{i+1}$, then it is called a *nilpotency series*. In this case, if $G_{k+1} = \{1\}$, then we say that G is *nilpotent of class* $\leq k$ and write nil $G \leq k$. Clearly, we have nil $G \geq$ solv G. In addition, we write the identity homomorphism of a group and the trivial homomorphism between two groups as $1\colon G \to G$ and $0\colon G \to H$, respectively. This notation is also used for sets with a distinguished element.

Finally, we fix our topological conventions and notation. By a *space*, we mean a connected CW-complex of finite type. Usually, we will be interested in finite-dimensional CW-complexes. When we discuss rational spaces, we will specialize to 1-connected CW-complexes. As is well known, such a space X admits a *rationalization*, which is denoted by $X_{\mathbb{Q}}$. Similarly, a map of 1-connected finite complexes $f\colon X \to Y$ admits a rationalization map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. A general reference for rationalization is [**HMR75**]. Furthermore, we do not distinguish notationally between a map and its homotopy class. We write $X \equiv Y$ to denote that the spaces X and Y have the same homotopy type. The identity map of a space X is denoted $\iota\colon X \to X$ and the trivial map between two spaces $*\colon X \to Y$.

2. A Connection with the Gottlieb Group

We consider the situation as in Section 1 of a mapping cone sequence

$$Z \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma Z,$$

and the induced action of $[\Sigma Z, W]$ on $[X, W]$ which yields $f^{\alpha} \in [X, W]$ for $\alpha \in [\Sigma Z, W]$ and $f \in [X, W]$. We are interested in the effect that f^{α} has on homology and homotopy groups. This is described in the following result.

PROPOSITION 2.1. *For the above cofibration sequence, suppose $f \in [X, W]$ and $\alpha \in [\Sigma Z, W]$.*

1. *The induced homology homomorphism $(f^{\alpha})_* \colon H_i(X) \to H_i(W)$ is given by $(f^{\alpha})_*(x) = f_*(x) + \alpha_* q_*(x)$, for each $x \in H_i(X)$.*
2. *Suppose that $(f, \alpha) \colon X \vee \Sigma Z \to W$ factors through the product $X \times \Sigma Z$. Then the induced homotopy homomorphism $(f^{\alpha})_{\#} \colon \pi_i(X) \to \pi_i(W)$ is given by $(f^{\alpha})_{\#}(x) = f_{\#}(x) + \alpha_{\#} q_{\#}(x)$, for each $x \in \pi_i(X)$.*

PROOF. (1) This follows directly from the commutative diagram

$$\begin{array}{ccccccc}
H_i(X) & \xrightarrow{c_*} & H_i(X \vee \Sigma Z) & \xrightarrow{(f \vee \alpha)_*} & H_i(W \vee W) & \xrightarrow{\nabla_*} & H_i(W) \\
& {\scriptstyle (1, q_*)} \searrow & \downarrow {\scriptstyle (p_{1*}, p_{2*})} & & \downarrow {\scriptstyle (p_{1*}, p_{2*})} & {\scriptstyle +} \nearrow & \\
& & H_i(X) \oplus H_i(\Sigma Z) & \xrightarrow[f_* \oplus \alpha_*]{} & H_i(W) \oplus H_i(W) & &
\end{array}$$

in which the vertical maps are isomorphisms induced by the two projections p_1 and p_2 and the top row is the homomorphism induced by f^{α}.

(2) Let $\sigma \colon S^i \to S^i \vee S^i$ denote the standard comultiplication. Write $f_{\#}(x) + \alpha_{\#} q_{\#}(x)$ as the composition

$$S^i \xrightarrow{(x \vee qx)\sigma} X \vee \Sigma Z \xrightarrow{(f, \alpha)} W$$

and $(f^{\alpha})_{\#}(x)$ as the composition

$$S^i \xrightarrow{cx} X \vee \Sigma Z \xrightarrow{(f, \alpha)} W .$$

By hypothesis, we can factor (f, α) through the product as $(f, \alpha) = a \circ j \colon X \vee \Sigma Z \to X \times \Sigma Z \to W$, for some $a \colon X \times \Sigma Z \to W$. It is straightforward to prove that $j(x \vee qx)\sigma = jcx \colon S^i \to X \times \Sigma Z$, by checking that their projections onto each summand are homotopic. $\qquad \square$

We now specialize to a mapping cone sequence of the form

$$S^{n-1} \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma S^{n-1} \equiv S^n,$$

i.e., $X = Y \cup_{\gamma} e^n$. Then we have an action of $\pi_n(X)$ on $[X, X]$. We consider elements of the form $\iota^{\alpha} \in [X, X]$, where ι is the identity map of X and $\alpha \in \pi_n(X)$. In general, these maps are not self-homotopy equivalences. However, by adding certain hypotheses, we obtain maps in $\mathcal{E}_{\#}(X)$, or some other subgroup of $\mathcal{E}(X)$. This approach is similar to that taken in [**AL91**], but instead of assuming $j_{\#} \colon \pi_{\#}(Y) \to \pi_{\#}(X)$ is onto, we shall consider restrictions on the homotopy element $\alpha \in \pi_n(X)$. Recall that the n'th *Gottlieb group* of X, denoted $G_n(X)$, consists of those $\alpha \in$

$\pi_n(X)$ for which there is an associated map $a\colon X \times S^n \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X \vee S^n & \xrightarrow{(\iota,\alpha)} & X\ . \\
{\scriptstyle j}\downarrow & \nearrow {\scriptstyle a} & \\
X \times S^n & &
\end{array}
$$

See [**Got69**] for various results on the groups $G_n(X)$.

Next, we introduce another subgroup of $\mathcal{E}(X)$. Define

$$\mathcal{E}_*(X) = \{f \in \mathcal{E}(X) \mid f_* = 1\colon H_i(X) \to H_i(X),\ \text{for all } i\}.$$

We apply Proposition 2.1 and obtain the following consequence.

COROLLARY 2.2. *Let* $X = Y \cup_\gamma e^n$ *be a* 1*-connected CW-complex and* $\alpha \in \pi_n(X)$.

1. $\iota^\alpha \in \mathcal{E}_*(X)$ *if and only if* $\alpha_* q_* = 0\colon H_n(X) \to H_n(X)$.
2. *Suppose* $\alpha \in G_n(X)$, *and* X *is of dimension* N. *Then* $\iota^\alpha \in \mathcal{E}_\#(X)$ *if and only if* $\alpha_\# q_\# = 0\colon \pi_i(X) \to \pi_i(X)$ *for* $i \leq N$.

PROOF. It is immediate from Proposition 2.1 that ι^α induces the identity on homology groups. Since X is a 1-connected CW-complex, ι^α is a homotopy equivalence. Hence $\iota^\alpha \in \mathcal{E}_*(X)$. This establishes (1), and (2) follows similarly. $\square$

Hence, we are interested in finding situations in which $\alpha q\colon X \to X$ induces the trivial homomorphism, either in homology or homotopy. Our first result is an integral result. Following this, we shall focus on the rational setting, where more information can be obtained.

THEOREM 2.3. *Let* $X = Y \cup_\gamma e^n$ *be a* 1*-connected* n*-dimensional complex. Suppose that* $q_\# = 0\colon \pi_n(X) \to \pi_n(S^n)$. *Then there is a homomorphism*

$$\Theta\colon G_n(X) \to \mathcal{E}_\#(X),$$

defined by $\Theta(\alpha) = \iota^\alpha$ *for* $\alpha \in G_n(X)$. *This homomorphism restricts to*

$$\Theta'\colon G_n(X) \cap \ker h_n \to \mathcal{E}_*(X) \cap \mathcal{E}_\#(X),$$

where $h_n\colon \pi_n(X) \to H_n(X)$ *denotes the Hurewicz homomorphism.*

PROOF. Let $\alpha \in G_n(X)$. Since $\pi_i(S^n) = 0$ for $i < n$, the hypothesis gives that $q_\# = 0\colon \pi_i(X) \to \pi_i(S^n)$ for $i \leq n$. Hence, by Corollary 2.2, $\iota^\alpha \in \mathcal{E}_\#(X)$. Now suppose β is any element in $\pi_n(X)$. Since $\iota^\alpha \in \mathcal{E}_\#(X)$ and $\beta \in \pi_n(X)$, we have that $\iota^\alpha(\beta) = \beta$. Thus, by the properties of the action listed in the introduction, we have

$$\iota^\alpha \iota^\beta = (\iota^\alpha)^{\iota^\alpha(\beta)} = \iota^{\alpha + \iota^\alpha(\beta)} = \iota^{\alpha+\beta}.$$

Therefore Θ is a homomorphism.

Now suppose α is any element in $\ker h_n$. Then $\alpha_*\colon H_n(S^n) \to H_n(X)$ is zero. Since $H_i(S^n) = 0$ for positive $i \neq n$, Corollary 2.2 implies that $\iota^\alpha \in \mathcal{E}_*(X)$. Thus Θ restricts to Θ' as claimed. $\square$

REMARK 2.4. It is known that $G_n(X) \subseteq \ker h_n$ under certain hypotheses (cf. [**Got69**, Th.4.1]), so the homomorphism Θ and its restriction Θ' may agree.

We illustrate Theorem 2.3 with an example.

EXAMPLE 2.5. Take $X = S^2 \times S^3 = S^2 \vee S^3 \cup_{[i_1, i_2]} e^5$. This kind of example has been considered previously (cf. [**AM98, Saw75**]), but here we put it into the context discussed above.

As is well-known, S^3 is an H-space and therefore satisfies $G_i(S^3) = \pi_i(S^3)$ for all i. Further, the Gottlieb group preserves products so $G_5(X) = G_5(S^2) \oplus G_5(S^3)$. Since $\pi_5(S^3) = \mathbb{Z}_2$, there is at least a non-trivial element of order 2 in $G_5(X)$. Next, consider $q_\# \colon \pi_5(X) \to \pi_5(S^5)$. Since $\pi_5(X) = \pi_5(S^3) \oplus \pi_5(S^2)$ is a finite group and $\pi_5(S^5)$ is infinite cyclic, it follows that $q_\#$ is zero in this dimension. From Theorem 2.3, Θ defines a homomorphism from $G_5(X)$ to $\mathcal{E}_\#(X)$. Notice that $h_5 \colon \pi_5(X) \to H_5(X)$ is zero, since $H_5(X)$ is infinite cyclic. Therefore, $G_5(X) \subseteq \ker h_5$ and $\Theta = \Theta' \colon G_5(X) \to \mathcal{E}_*(X) \cap \mathcal{E}_\#(X)$.

Note that the homomorphism Θ in Theorem 2.3 may have trivial image in $\mathcal{E}_\#(X)$. For Example 2.5, it follows from the computations in [**AM98**, §6] that Θ is actually injective. However, it seems to be difficult to give general conditions to guarantee that Θ is injective. Rather than doing this by placing strong hypotheses on our spaces, we turn now to the rational setting. For a 1-connected CW-complex X, the *rational Gottlieb group* of X is the Gottlieb group of the rationalization of X, that is, $G_n(X_\mathbb{Q})$. Notice that by [**Lan75**], we have $G_n(X_\mathbb{Q}) \cong G_n(X) \otimes \mathbb{Q}$ for each n. In contrast to the ordinary Gottlieb groups, much is known about the rational Gottlieb groups by results of Félix-Halperin [**FH82**]. For instance, a 1-connected, finite complex has no non-trivial rational Gottlieb groups of even degree, and has only finitely many non-trivial rational Gottlieb groups of odd degree (see [**Fél89**] for details). We only touch on these ideas here and avoid heavy use of rational techniques.

LEMMA 2.6. *Suppose we have a mapping cone sequence*

$$S^{n-1} \xrightarrow{\gamma} Y \xrightarrow{j} X$$

in which Y and X are 1-connected. If $\gamma_\mathbb{Q} \neq * \colon (S^{n-1})_\mathbb{Q} \to Y_\mathbb{Q}$, then $(j_\mathbb{Q})_\# \colon \pi_n(Y_\mathbb{Q}) \to \pi_n(X_\mathbb{Q})$ is surjective.

PROOF. This can be argued using the long exact homotopy and homology sequences, together with the relative Hurewicz theorem, for the pair (X, Y). Alternatively, Quillen minimal models can be used. We omit the details. □

REMARK 2.7. Notice we assert that $(j_\mathbb{Q})_\#$ is onto in degree n only, and not in all degrees. In the latter case, the cell attachment is called an *inert* cell attachment [**HL87**]. This is one of the hypotheses used in [**AL91**], but it is not satisfied by some of the examples we have in mind.

We will see that under our hypotheses, rational equivalences of the form ι^α are contained in a smaller subgroup of $\mathcal{E}(X)$ than $\mathcal{E}_\#(X)$. We introduce the following notation: For $r \leq \infty$, define

$$\mathcal{E}_{\#r}(X) = \{f \in \mathcal{E}(X) \mid f_\# = 1 \colon \pi_i(X) \to \pi_i(X), \text{ for all } i \leq r\}.$$

Note that $f \in \mathcal{E}_{\#\infty}(X)$ if and only if f induces the identity homomorphism of *all* homotopy groups.

The following is our basic rational result.

THEOREM 2.8. *Let $X = Y \cup_\gamma e^n$ be a 1-connected CW complex with n odd and $\gamma_\mathbb{Q} \neq *\colon (S^{n-1})_\mathbb{Q} \to Y_\mathbb{Q}$. Then there is a homomorphism*

$$\Phi\colon G_n(X_\mathbb{Q}) \to \mathcal{E}_{\#\infty}(X_\mathbb{Q}),$$

defined by $\Phi(\alpha) = \iota^\alpha$ for $\alpha \in G_n(X_\mathbb{Q})$. This homomorphism restricts to

$$\Phi'\colon G_n(X_\mathbb{Q}) \cap \ker h_n \to \mathcal{E}_*(X_\mathbb{Q}) \cap \mathcal{E}_{\#\infty}(X_\mathbb{Q}),$$

where h_n denotes the rational Hurewicz homomorphism $h_n\colon \pi_n(X_\mathbb{Q}) \to H_n(X_\mathbb{Q})$.

PROOF. We proceed as in the proof of Theorem 2.3. First, we claim that $(q_\mathbb{Q})_\# = 0\colon \pi_i(X_\mathbb{Q}) \to \pi_i(S^n_\mathbb{Q})$, for all i. Since n is odd, we have $\pi_i(S^n_\mathbb{Q}) = 0$ for $i \neq n$. Hence we must only check that $(q_\mathbb{Q})_\# = 0$ in degree n. By Lemma 2.6, $(j_\mathbb{Q})_\#\colon \pi_n(Y_\mathbb{Q}) \to \pi_n(X_\mathbb{Q})$ is surjective. Given $x \in \pi_n(X_\mathbb{Q})$, write $x = (j_\mathbb{Q})_\#(y)$, for some $y \in \pi_n(Y_\mathbb{Q})$. Then $(q_\mathbb{Q})_\#(x) = (q_\mathbb{Q})_\#(j_\mathbb{Q})_\#(y) = ((qj)_\mathbb{Q})_\#(y) = 0$ since $qj = *$, and the claim follows.

Now a simple modification of the proof of Lemma 2.2 yields that $\iota^\alpha \in \mathcal{E}_{\#\infty}(X_\mathbb{Q})$, for each $\alpha \in G_n(X_\mathbb{Q})$. The remainder of the argument follows exactly as in the proof of Theorem 2.3. $\qquad\square$

REMARK 2.9. Notice that, unlike Theorem 2.3, there is no restriction on the dimension of X in Theorem 2.8, and the attached cell need not be top-dimensional. If X is a 1-connected, finite complex, then there is no generality lost in assuming n odd since a 1-connected, finite complex has no non-trivial rational Gottlieb groups of even degree.

Although Theorem 2.8 is a rational result, we are able to 'de-rationalize' it to obtain the following integral consequence.

THEOREM 2.10. *Let $X = Y \cup_\gamma e^n$ be a 1-connected finite complex with n odd and $\gamma \in \pi_{n-1}(Y)$ not of finite order. If the homomorphism Φ from Theorem 2.8 is non-zero, then for each r with $\dim X \leq r < \infty$, there are elements of infinite order in $\mathcal{E}_{\#r}(X)$.*

PROOF. Suppose $\Phi(\alpha) = \iota^\alpha$ is not the identity element in $\mathcal{E}_{\#\infty}(X_\mathbb{Q})$. Since this latter is a $\mathbb{Q}$-local group, it contains no non-trivial elements of finite order, and hence $(\iota^\alpha)^k \neq \iota\colon X_\mathbb{Q} \to X_\mathbb{Q}$, for all k. By [**Mar89**], we have $\mathcal{E}_{\#r}(X_\mathbb{Q}) \cong (\mathcal{E}_{\#r}(X))_\mathbb{Q}$, for each r with $\dim X \leq r < \infty$. From this it follows that for each r there is some positive integer p and some element $f \in \mathcal{E}_{\#r}(X)$ such that $f_\mathbb{Q} = (\iota^\alpha)^p$. Since $f_\mathbb{Q}$ is of infinite order in $\mathcal{E}_{\#r}(X_\mathbb{Q})$, the same must be true of f in $\mathcal{E}_{\#r}(X)$. $\qquad\square$

3. Solvability of $\mathcal{E}_\#(Y)$

A result of Dror-Zabrodsky asserts that if Y is a finite complex, then $\mathcal{E}_\#(Y)$ is a nilpotent group [**DZ79**]. One can ask, therefore, whether there are reasonable estimates for the nilpotency, or perhaps the solvability, of $\mathcal{E}_\#(Y)$ in terms of the usual algebraic topological invariants of Y. Several results have been established that relate the nilpotency or solvability of $\mathcal{E}_\#(Y)$, or some similar group, to the *Lusternik-Schnirelmann category* of Y, or related invariants (cf. [**AL96, FM97, FM98, ST99**]). Some of these apply in a rational setting, and others in an integral setting. Typically, these results give an upper bound on the nilpotency or solvability of the group.

We begin by discussing a topological invariant which appears in our results.

DEFINITION 3.1. For any space X, a *spherical cone decomposition of X of length n*, is a sequence of cofibrations

$$L_i \xrightarrow{\gamma_i} X_i \xrightarrow{j_i} X_{i+1},$$

for $0 \leq i < n$, such that each L_i is a finite wedge of spheres, X_0 is contractible and $X_n \equiv X$. We define the *spherical cone-length* of X, denoted by $\mathrm{scl}(X)$, as follows: If X is contractible, then set $\mathrm{scl}(X) = 0$. Otherwise, $\mathrm{scl}(X)$ is the smallest positive integer n such that there exists a spherical cone decomposition of X of length n. If no such integer exists, set $\mathrm{scl}(X) = \infty$. If X is a finite-dimensional complex and we have a spherical cone decomposition of X of length n in which, in addition, $\dim L_i < \dim X$ for $i = 0, \dots, n-1$, then this is called a *restricted spherical cone decomposition of X of length n*. We then define the *restricted spherical cone-length* of X, denoted $\mathrm{rscl}(X)$, using only restricted spherical cone decompositions in place of ordinary spherical cone decompositions.

REMARK 3.2. Spherical cone-length has been considered in [**Cor94, ST99**]. If we denote the Lusternik-Schnirelmann category of X by $\mathrm{cat}(X)$, then it is known that $\mathrm{cat}(X) \leq \mathrm{scl}(X)$. Note that a space X with $\mathrm{scl}(X) = 1$ is homotopy equivalent to a wedge of spheres and that a space X with $\mathrm{scl}(X) \leq 2$ is homotopy equivalent to the cofibre of a map between wedges of spheres. Furthermore, the cell-structure of a finite-dimensional complex X provides a restricted spherical cone decomposition of length $\leq$ the number of dimensions in which there are positive-dimensional cells.

We introduce a bit more notation before proving the main result of this section. Once again, Y is a complex of dimension N. We define

$$\mathcal{E}_k(Y) = \{f \in \mathcal{E}(Y) \mid f_* = 1 \colon [X,Y] \to [X,Y], \text{ for every complex } X$$
$$\text{with } \dim X \leq N \text{ and } \mathrm{rscl}(X) \leq k\}.$$

In particular, we have $\mathcal{E}_1(Y) = \mathcal{E}_\#(Y)$. Also, there is a chain of subgroups

$$(1) \qquad \mathcal{E}_\#(Y) = \mathcal{E}_1(Y) \supseteq \mathcal{E}_2(Y) \supseteq \cdots \supseteq \mathcal{E}_k(Y) \supseteq \cdots .$$

Clearly we have $\mathcal{E}_k(Y) \triangleleft \mathcal{E}_{k-1}(Y)$: For if $f \in \mathcal{E}_k(Y)$, $g \in \mathcal{E}_{k-1}(Y)$, $\dim X \leq N$ and $\mathrm{rscl}(X) \leq k$, then $f_* g_*^{-1} = g_*^{-1} \colon [X,Y] \to [X,Y]$. Hence

$$(gfg^{-1})_* = g_* f_* g_*^{-1} = g_* g_*^{-1} = 1,$$

and so $gfg^{-1} \in \mathcal{E}_k(Y)$. Therefore, the series (1) is a normal chain. Furthermore, if $\mathrm{rscl}(Y) \leq k$, then $\mathcal{E}_k(Y) = 1$. Then we have a normal chain

$$\mathcal{E}_\#(Y) \supseteq \mathcal{E}_2(Y) \supseteq \cdots \supseteq \mathcal{E}_k(Y) = \{1\}.$$

The following is the main result of this section.

THEOREM 3.3. *The series (1) is a solvability series, i.e., $[\mathcal{E}_i(Y), \mathcal{E}_i(Y)] \subseteq \mathcal{E}_{i+1}(Y)$. Consequently, we have*

$$\mathrm{solv}\, \mathcal{E}_\#(Y) \leq \mathrm{rscl}(Y) - 1.$$

PROOF. Let $f, g \in \mathcal{E}_i(Y)$ and X be a complex with $\mathrm{rscl}(X) = i+1$ and $\dim X \leq N$. It suffices to show that $f_* g_*(h) = g_* f_*(h)$ for every $h \in [X,Y]$. Consider the last cofibre sequence in a length-$(i+1)$ restricted spherical cone decomposition of X,

$$L_i \xrightarrow{\gamma} X_i \xrightarrow{j} X_{i+1},$$

where L_i is a wedge of spheres and $X_{i+1} \equiv X$. Now, since $f \in \mathcal{E}_i(Y)$, it follows that $j^*(fh) = f_*(hj) = j^*(h)$. Thus, from the properties of the coaction reviewed in Section 1, there is some $\alpha \in [\Sigma L_i, Y]$ such that $fh = h^\alpha$. Similarly, there is some $\beta \in [\Sigma L_i, Y]$ such that $g_*(h) = h^\beta$. Note also that $f\beta = \beta$ since $f \in \mathcal{E}_i(Y) \subseteq \mathcal{E}_1(Y)$, and ΣL_i is a wedge of spheres of dimension $\leq N$. Now we have

$$f_* g_*(h) = f(h^\beta) = (fh)^{f\beta} = (h^\alpha)^\beta = h^{\alpha+\beta}.$$

A similar computation yields $g_* f_*(h) = h^{\beta+\alpha}$. Since $[\Sigma L_i, Y]$ is abelian, the proof is complete. $\square$

REMARK 3.4. A result analogous to Theorem 3.3 has been proved by Scheerer and Tanré in [**ST99**, Th.6]. We note the differences and similarities between these results. Theorem 6 in [**ST99**] is proved for the group of equivalences of a space Y relative to certain fixed classes of spaces (though our proof could be easily modified to hold for these classes). When the class consists of wedges of spheres, the corresponding group of equivalences is $\mathcal{E}_{\#\infty}(Y)$. The upper bound for the solvability of the group of equivalences $\mathcal{E}_{\#\infty}(Y)$ relative to the class of wedges of spheres given in [**ST99**] is then the so-called spherical category of Y, which is less than or equal to the spherical cone-length of Y minus one. On the other hand, the group of equivalences $\mathcal{E}_\#(Y)$ that we consider in Theorem 3.3 is larger than $\mathcal{E}_{\#\infty}(Y)$. Furthermore, the two proofs are similar, but the solvability series in Theorem 3.3 appears to be different from the one in [**ST99**, Th.6].

Theorem 3.3 easily gives the next two corollaries.

COROLLARY 3.5. *If* $\mathrm{rscl}(Y) \leq 2$, *that is,* Y *is the cofibre of a map between wedges of spheres, then* $\mathcal{E}_\#(Y)$ *is abelian.*

COROLLARY 3.6. *For* Y *any finite-dimensional complex,* $\mathcal{E}_\#(Y)$ *is solvable, with* $\mathrm{solv}\,\mathcal{E}_\#(Y) \leq k - 1$, *where* k *is the number of dimensions in which there are positive-dimensional cells.*

PROOF. This follows by Remark 3.2. $\square$

We can modify much of the previous material to deal with equivalences that fix all homotopy groups. This also allows us to deal with the case in which Y is an arbitrary space which is not necessarily a finite-dimensional complex. Define

$$\mathcal{E}'_k(Y) = \{f \in \mathcal{E}(Y) \mid f_* = 1 \colon [X, Y] \to [X, Y], \text{ for all } X \text{ with } \mathrm{scl}(X) \leq k\}.$$

Then there is a normal chain

$$(2) \qquad \mathcal{E}_{\#\infty}(Y) = \mathcal{E}'_1(Y) \supseteq \mathcal{E}'_2(Y) \supseteq \cdots \supseteq \mathcal{E}'_k(Y) \supseteq \cdots.$$

Now the proof of Theorem 3.3 yields the following analogous results.

THEOREM 3.7. [**ST99**, Th. 6] *The series (2) is a solvability series. Therefore,*

$$\mathrm{solv}\,\mathcal{E}_{\#\infty}(Y) \leq \mathrm{scl}(Y) - 1.$$

We note that from a spectral sequence of Didierjean [**Did85**], one can also obtain a different upper bound on the solvability of $\mathcal{E}_{\#\infty}(Y)$ in terms of the cohomology of Y with coefficients in the homotopy groups of Y.

COROLLARY 3.8. *If* $\mathrm{scl}(Y) \leq 2$, *then* $\mathcal{E}_{\#\infty}(Y)$ *is abelian.*

In view of the results in this section, together with the bounds found in [**AL96, FM97, FM98, ST99**], it is natural to believe that, for a finite-dimensional complex Y, the nilpotency class of $\mathcal{E}_\#(Y)$ is bounded above by $\mathrm{scl}(Y) - 1$. We have not been able to prove this, and so we leave it as a conjecture.

CONJECTURE 3.9. *For a finite-dimensional complex Y,*

$$\mathrm{nil}\,\mathcal{E}_\#(Y) \leq \mathrm{rscl}(Y) - 1 \qquad and \qquad \mathrm{nil}\,\mathcal{E}_{\#\infty}(Y) \leq \mathrm{scl}(Y) - 1.$$

We note that Scheerer and Tanré have conjectured that $\mathrm{nil}\,\mathcal{E}_{\#\infty}(Y)$ is bounded above by the spherical category of Y [**ST99**, §7, (6)].

Conjecture 3.9 would be established by showing that each of the series (1) and (2) is a nilpotency series. A direct proof of this would also give an independent proof of the Dror-Zabrodsky result on the nilpotency of $\mathcal{E}_\#(Y)$.

4. Equivalences that Fix Cohomology Groups

In this section, we dualize some of the ideas of the previous section. Although we did not use homotopy groups with coefficients there, we do use coefficients here, since this is more common with cohomology.

Let $\mathcal{G}$ be a collection of abelian groups and X be a space. Define

$$\mathcal{E}_\mathcal{G}^*(X) = \{f \in \mathcal{E}(X) \mid f^* = 1\colon H^i(X;G) \to H^i(X;G), \text{ for all } i \text{ and all } G \in \mathcal{G}\}.$$

The following cases are of special interest:

1. $\mathcal{G} = \{\mathbb{Z}\}$. We write $\mathcal{E}_\mathcal{G}^*(X)$ as $\mathcal{E}^*(X)$ in this case.
2. $\mathcal{G} = $ all cyclic groups. Then $f \in \mathcal{E}_\mathcal{G}^*(X)$ if and only if $f \in \mathcal{E}(X)$ and $f^* = 1\colon H^i(X;G) \to H^i(X;G)$ for every finitely-generated abelian group G. We write $\mathcal{E}_\mathcal{G}^*(X)$ as $\mathcal{E}_{\mathrm{fg}}^*(X)$ in this case. Note that $\mathcal{E}_{\mathrm{fg}}^*(X) \subseteq \mathcal{E}^*(X)$.

Next we define a topological invariant that plays a role dual to that of spherical cone-length in the previous section.

DEFINITION 4.1. For $\mathcal{G}$ a collection of abelian groups, call an Eilenberg-Mac-Lane space $K(G, m)$, with $G \in \mathcal{G}$, a $\mathcal{G}$-*Eilenberg-MacLane space*. For any space X, a $\mathcal{G}$-*fibre decomposition of X of length n*, is a sequence of fibrations

$$X_{i+1} \xrightarrow{\ j_i\ } X_i \xrightarrow{\ p_i\ } K_i,$$

for $0 \leq i < n$, such that each K_i is a finite product of $\mathcal{G}$-Eilenberg-MacLane spaces, X_0 is contractible and $X_n \equiv X$. We define the $\mathcal{G}$-*fibre-length* of X, denoted by $\mathcal{G}$-$\mathrm{fl}(X)$, by dualizing Definition 3.1 in a straightforward way.

Note that a space X with $\mathcal{G}$-$\mathrm{fl}(X) = 1$ is homotopy equivalent to a product of $\mathcal{G}$-Eilenberg-MacLane spaces. A space X with $\mathcal{G}$-$\mathrm{fl}(X) \leq 2$ is homotopy equivalent to the fibre of a map between products of $\mathcal{G}$-Eilenberg-MacLane spaces. Note also that when we mention a product of $\mathcal{G}$-Eilenberg-MacLane spaces, we allow factors with homotopy groups in different dimensions, so that a product of $\mathcal{G}$-Eilenberg-MacLane spaces is not itself a $\mathcal{G}$-Eilenberg-MacLane space in general.

Now define subgroups of $\mathcal{E}(X)$ as follows:

$$\mathcal{E}_{\mathcal{G},s}^*(X) = \{f \in \mathcal{E}(X) \mid f^* = 1\colon [X, Y] \to [X, Y], \text{ for all } Y \text{ with } \mathcal{G}\text{-}\mathrm{fl}(Y) \leq s\}.$$

Then there is a normal chain of subgroups

$$(3) \qquad \mathcal{E}_\mathcal{G}^*(X) = \mathcal{E}_{\mathcal{G},1}^*(X) \supseteq \mathcal{E}_{\mathcal{G},2}^*(X) \supseteq \cdots \supseteq \mathcal{E}_{\mathcal{G},s}^*(X) \supseteq \cdots.$$

The proof of normality for (3) is similar to the proof of normality for (1) above.

A straightforward dualization of the proof of Theorem 3.3 yields the following result.

THEOREM 4.2. *The series (3) is a solvability series. Thus*

$$\text{solv}\,\mathcal{E}_{\mathcal{G}}^*(X) \leq \mathcal{G}\text{-fl}(X) - 1.$$

In Corollary 3.6, we showed that the number of dimensions in which Y has cells may be used to estimate the spherical cone length. We now indicate briefly how the preceding notions can be modified for the dual result.

DEFINITION 4.3. A space X is called a *Postnikov piece* if there is some N such that $\pi_i(X) = 0$ for all $i > N$. The smallest such N is called the *homotopical dimension* of X, and is denoted h-dim X.

For a Postnikov piece X, we define another subgroup of $\mathcal{E}(X)$ as

$$\mathcal{E}_{\mathcal{G}}^{*\prime}(X) = \{f \in \mathcal{E}(X) \mid f^* = 1 \colon H^i(X;G) \to H^i(X;G), \text{ for all } i \leq \text{h-dim}\,X$$
$$\text{and all } G \in \mathcal{G}\}.$$

We then define a *restricted $\mathcal{G}$-fibre decomposition* of a Postnikov piece X, of length n, as above but with the additional condition that h-dim $K_i \leq$ h-dim $X + 1$, for all i. This yields the *restricted $\mathcal{G}$-fibre-length* of X, denoted by r-$\mathcal{G}$-fl(X).

REMARK 4.4. Let X be a 1-connected space and $X^{(N)}$ be the N'th Postnikov section of X. Set $\mathcal{G} =$ all cyclic groups and $s =$ the number of non-trivial homotopy groups of $X^{(N)}$. Then by taking the Postnikov decomposition of $X^{(N)}$, we see that r-$\mathcal{G}$-fl$(X^{(N)}) \leq s$.

Now if X is a Postnikov piece, define

$$\mathcal{E}_{\mathcal{G},s}^{*\prime}(X) = \{f \in \mathcal{E}(X) \mid f^* = 1 \colon [X,Y] \to [X,Y], \text{ for all Postnikov pieces } Y$$
$$\text{such that h-dim}\,Y \leq \text{h-dim}\,X \text{ and r-}\mathcal{G}\text{-fl}(Y) \leq s\}.$$

Then once again we have a normal chain of subgroups

$$(4) \qquad \mathcal{E}_{\mathcal{G}}^{*\prime}(X) = \mathcal{E}_{\mathcal{G},1}^{*\prime}(X) \supseteq \mathcal{E}_{\mathcal{G},2}^{*\prime}(X) \supseteq \cdots \supseteq \mathcal{E}_{\mathcal{G},s}^{*\prime}(X) \supseteq \cdots.$$

Here also, the proof of normality is similar to the proof of normality for (1) above. A further adaptation of the proof of Theorem 3.3 gives the following result.

THEOREM 4.5. *The series (4) is a solvability series. Thus ,*

$$\text{solv}\,\mathcal{E}_{\mathcal{G}}^{*\prime}(X) \leq \text{r-}\mathcal{G}\text{-fl}(X) - 1,$$

for a Postnikov piece X.

This leads to the dual of Corollary 3.6.

COROLLARY 4.6. *If X is a 1-connected finite complex, then $\mathcal{E}_{\text{fg}}^*(X)$ is solvable. In particular, if X has dimension N and there are s non-trivial homotopy groups in degrees $\leq N$, then* solv $\mathcal{E}_{\text{fg}}^*(X) \leq s - 1$.

PROOF. Any map $f \colon X \to X$ induces a corresponding map $\theta(f) \colon X^{(N)} \to X^{(N)}$ of N'th Postnikov sections. This gives us a homomorphism $\theta \colon \mathcal{E}_{\mathcal{G}}^*(X) \to \mathcal{E}_{\mathcal{G}}^{*\prime}(X^{(N)})$ which is one-one. Therefore, solv $\mathcal{E}_{\mathcal{G}}^*(X) \leq$ solv $\mathcal{E}_{\mathcal{G}}^{*\prime}(X^{(N)})$. But if $\mathcal{G}$ is the collection of all cyclic groups, then it is a consequence of Remark 4.4 and Theorem 4.5 that solv $\mathcal{E}_{\mathcal{G}}^{*\prime}(X^{(N)}) \leq s - 1$. □

Next, we compare the subgroups $\mathcal{E}^*(X)$ and $\mathcal{E}^*_{\text{fg}}(X)$. Below, we give a simple example to illustrate that these two subgroups are distinct in general. First, however, we obtain conditions under which they agree.

PROPOSITION 4.7. *For any space X, if*

$$\text{Hom}\big(\text{Tor}(H^{i+1}(X), G), H^i(X) \otimes G\big) = 0$$

for all i and all finitely-generated groups G, then $\mathcal{E}^*(X) = \mathcal{E}^*_{\text{fg}}(X)$.

PROOF. If $f \in \mathcal{E}^*(X)$, then the universal coefficient theorem (cf. [**Spa66**, Thm. 5.10]) gives a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^i(X) \otimes G & \overset{\iota}{\longrightarrow} & H^i(X; G) & \overset{\pi}{\longrightarrow} & \text{Tor}(H^{i+1}(X), G) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & H^i(X) \otimes G & \overset{\iota}{\longrightarrow} & H^i(X; G) & \overset{\pi}{\longrightarrow} & \text{Tor}(H^{i+1}(X), G) & \longrightarrow & 0
\end{array}
$$

where the middle homomorphism ϕ can be either f^* or the identity 1. Thus there is a homomorphism $\rho \colon \text{Tor}(H^{i+1}(X), G) \to H^i(X) \otimes G$ such that $f^* - 1 = \iota \rho \pi$. By hypothesis, $\rho = 0$ and so $f^* = 1$. Therefore, $f \in \mathcal{E}^*_{\text{fg}}(X)$. $\qquad\square$

The following example illustrates that $\mathcal{E}^*(X)$ and $\mathcal{E}^*_{\text{fg}}(X)$ may differ.

EXAMPLE 4.8. Let X be a Moore space $M(G, n)$, for $n \geq 2$ and G any infinite, finitely-generated abelian group with torsion. Then it follows from results of [**AM98**] that $\mathcal{E}^*(X) \neq \mathcal{E}^*_{\text{fg}}(X)$

In Corollary 4.6, we showed that for a 1-connected, finite-dimensional complex X, $\mathcal{E}^*_{\text{fg}}(X)$ is solvable. This raises the question of whether or not the group is nilpotent. We conclude the paper by showing that $\mathcal{E}^*(X)$, and therefore $\mathcal{E}^*_{\text{fg}}(X)$, is a nilpotent group. We will use the following notation: Suppose that a group G acts on an abelian group A. We define inductively a decreasing sequence of subgroups of A by setting $\Gamma_1^G(A) = A$ and $\Gamma_i^G(A)$ is the subgroup generated by $\{ga - a \mid g \in G, \, a \in \Gamma_{i-1}^G(A)\}$. We say that the *action is nilpotent* if, for some i, $\Gamma_i^G(A) = \{0\}$.

PROPOSITION 4.9. *For any nilpotent finite complex X, $\mathcal{E}^*(X)$ is a nilpotent group.*

PROOF. We shall prove that $\mathcal{E}^*(X)$ acts nilpotently on $H_*(X)$. Then it follows from [**DZ79**] that $\mathcal{E}^*(X)$ is a nilpotent group.

Let $f \in \mathcal{E}^*(X)$ and consider the diagram with exact rows obtained from the universal coefficient theorem

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ext}(H_{*-1}(X), \mathbb{Z}) & \longrightarrow & H^*(X) & \longrightarrow & \text{Hom}(H_*(X), \mathbb{Z}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \text{Ext}(f_*, \mathbb{Z})} & & \downarrow{\scriptstyle f^* = 1} & & \downarrow{\scriptstyle \text{Hom}(f_*, \mathbb{Z})} & & \\
0 & \longrightarrow & \text{Ext}(H_{*-1}(X), \mathbb{Z}) & \longrightarrow & H^*(X) & \longrightarrow & \text{Hom}(H_*(X), \mathbb{Z}) & \longrightarrow & 0.
\end{array}
$$

Then both $\text{Ext}(f_*, \mathbb{Z})$ and $\text{Hom}(f_*, \mathbb{Z})$ are identity maps. Write $H_*(X) = F \oplus T$ as the sum of its free and torsion parts and let $p_T \colon F \oplus T \to T$ and $p_F \colon F \oplus T \to F$ be the projections. Since $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m$, it follows that $\text{Ext}(H_*(X), \mathbb{Z}) = \text{Ext}(T, \mathbb{Z}) = T$. Thus $\text{Ext}(f_*, \mathbb{Z}) = 1$ implies that $p_T \circ f_*|_T \colon T \to T$ is the identity.

In the same way, since $\mathrm{Hom}(f_*, \mathbb{Z}) = 1$, we have that $p_F \circ f_*|_F : F \to F$ is also the identity. Therefore, $f_* : F \oplus T \to F \oplus T$ can be written as $f_*(x, y) = (x, y + \phi(x))$, with $\phi : F \to T$ a homomorphism that depends on f. Hence $\Gamma_1^{\mathcal{E}^*(X)}(H_*(X))$ is generated by elements of the form $f_*(x, y) - (x, y) = (0, \phi(x))$. On these elements, any $g \in \mathcal{E}^*(X)$ satisfies $g_*(0, \phi(x)) = (0, \phi(x))$. Hence $\Gamma_2^{\mathcal{E}^*(X)}(H_*(X)) = \{0\}$, and the result follows. $\qquad\square$

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